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BERGSHOEFF, E; HULL, C; ORTIN, T

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Duality in the type-II superstring effective action

Eric Bergshoeff^a, Chris Hull^b, Tomás Ortín^{b,1}

^a *Institute for Theoretical Physics, University of Groningen, Nijenborgh 4, 9747 AG Groningen, The Netherlands*

^b *Department of Physics, Queen Mary and Westfield College, Mile End Road, London E1 4NS, United Kingdom*

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Abstract

We derive the T -duality transformations that transform a general $d = 10$ solution of the type-IIA string with one isometry to a solution of the type-IIB string with one isometry and vice versa. In contrast to other superstring theories, the T -duality transformations are not related to a non-compact symmetry of a $d = 9$ supergravity theory. We also discuss S -duality in $d = 9$ and $d = 10$ and the relationship with eleven-dimensional supergravity theory. We apply these dualities to generate new solutions of the type-IIA and type-IIB superstrings and of eleven-dimensional supergravity.

1. Introduction

Duality symmetries [1–5] play an important role in string theories and it has recently been found that duality symmetries of type-II strings have a number of interesting and unusual features [3]. The aim of this paper is to explore duality symmetries and some of their applications in the context of the type-II string in nine and ten dimensions, and the relation of these to eleven-dimensional supergravity. In particular, we aim to understand the T -duality symmetry of the type-II string in backgrounds with one isometry. This symmetry is of a rather unusual type in that it maps type-IIA backgrounds into type-IIB ones, and vice versa [6,7]. Moreover, whereas in the heterotic string T -duality for backgrounds with one isometry can be understood as a symmetry of nine-dimensional $N = 1$ supergravity, no such understanding is possible here: the type-II T -duality does

¹ Address after October 1st: Theory Division, CERN, CH-1211, Geneva 23, Switzerland.

not correspond to any symmetry of the nine-dimensional $N = 2$ supergravity theory. A discussion of our results has been given recently by one of us [8], and there is some overlap with the results of Witten [9] announced at the same conference.

The bosonic string compactified from $D + d$ dimensions to D dimensions on a d -torus T^d has an $O(d, d)$ duality symmetry which is broken to the discrete subgroup $O(d, d; \mathbb{Z})$ by non-perturbative sigma-model effects. (Either $D + d = 26$, or there is an additional hidden sector describing internal degrees of freedom through a CFT with $c = 26 - D - d$, which is suppressed in the following.) This discrete target-space duality or T-duality group includes the well-known $R \rightarrow \alpha'/R$ duality for each circle in T^d , where R is the radius, together with shifts of the antisymmetric tensor gauge field and $O(d; \mathbb{Z})$ rotations of the circles into one another; the latter are particular $(D + d)$ -dimensional diffeomorphisms. The $O(d, d; \mathbb{Z})$ is a discrete gauge group, and configurations related by such a duality transformation are physically equivalent. The $O(d, d)$ group is not a string symmetry, but transforms a consistent string background to a new one.

This can be generalized to consider the string on a curved $(D + d)$ -dimensional space with d commuting isometries. For consistency, the background must define a conformally invariant sigma-model which implies that the background fields must satisfy certain field equations, which can be derived from a low-energy effective action. There is again an $O(d, d)$ symmetry transforming solutions of the low-energy equations of motion into new ones; this was first shown for $d = 1$ by Buscher [10] and generalized to higher d in Ref. [11]. If the orbits of the isometries are compact, then there is again a discrete subgroup $O(d, d; \mathbb{Z})$ which is a discrete gauge symmetry of the string theory, but there will be Buscher duality even for non-compact orbits. Although the name T-duality is usually reserved for the discrete group $O(d, d; \mathbb{Z})$, we shall refer here also to the $O(d, d)$ Buscher group as a T-duality.

For the heterotic superstring without background gauge fields and the type-II superstring without background fields from the Ramond-Ramond (RR) sector, the situation is similar. For such backgrounds with d commuting isometries, the arguments of Refs. [10,11] give an $O(d, d)$ T-duality symmetry of the equations of motion and an $O(d, d; \mathbb{Z})$ discrete gauge symmetry if the orbits are compact. For the heterotic string, including sixteen background Abelian gauge fields enlarges the T-duality group to $O(d, d + 16)$. The main aim here will be to study duality in the type-II string in the presence of background RR fields. In particular, we shall be interested in ten-dimensional type-II backgrounds with one isometry. For a string moving in the special background $M^9 \times T^1$ (nine-dimensional Minkowski space times a one-torus) this has already been done in Refs. [6,7]. In these references it was argued that there is a \mathbb{Z}_2 T-duality symmetry that relates the type-IIA string on a circle of radius R and type-IIB string moving on a circle of radius α'/R . We will find a generalization of Buscher's transformation that transforms a solution of the type-IIA string on a background with one isometry to a solution of the type-IIB string on a background with one isometry. The transformation is essentially that of Buscher when restricted to fields in the Neveu-Schwarz/Neveu-Schwarz (NS-NS) sector, but interchanges the RR background fields of the type-IIA string with those of the type-IIB string. We shall use such transformations to generate

new solutions of the equations of motion. Since this type-II T-duality maps all the fields of the type-IIA string into the type-IIB and vice versa, it also maps the symmetries and can be used, for instance, to find in the type-IIA theory with one isometry the form of the $SL(2, \mathbb{R})$ -duality transformations which are well-known in the type-IIB theory.

Duality symmetries of the full string theory necessarily give rise to symmetries of the low-energy effective supergravity theory. In this paper, we shall study duality symmetries of such effective supergravity theories to lowest order in α' , which constitutes a first step toward studying string dualities. In addition to the T-dualities which are perturbative string symmetries and so can be studied using world-sheet sigma-models, there are also symmetries of the supergravity actions which correspond to conjectured non-perturbative S- and U- duality string symmetries. In particular, the S-duality group includes transformations which act on the dilaton $\hat{\phi}$. The type-IIB supergravity in ten dimensions has an explicit $SL(2, \mathbb{R})$ symmetry of the equations of motion which is broken to $SL(2, \mathbb{Z})$ by quantum corrections. This is the conjectured $SL(2, \mathbb{Z})$ S-duality discrete gauge symmetry of the type-IIB string [3], while $SL(2, \mathbb{R})$, which we shall also refer to as an S-duality, is a solution-generating symmetry, transforming any given solution into a new one. The type-IIA has an $SO(1,1)$ S-duality symmetry which acts rather trivially through a shift of the dilaton and scaling of the other fields. There is only one kind of $N = 2$ supergravity in nine dimensions, so that compactifying either type-IIA or type-IIB supergravities to nine (or less) dimensions gives the same compactified theory, which inherits the symmetries of both of its two parent theories. The nine-dimensional theory has an $SL(2, \mathbb{R})$ symmetry, which is broken to the $SL(2, \mathbb{Z})$ S-duality by quantum effects [3]. The $SL(2, \mathbb{R})$ can be thought of as arising from the $SL(2, \mathbb{R})$ symmetry of the type-IIB theory, but its origins from the type-IIA theory are not so clear. In [12] a relation between the type-IIA string and the 11-dimensional membrane compactified to $d = 10$ on a circle was suggested in which the dilaton emerges as a modulus field for the compact dimension. We shall find further evidence for the role of eleven dimensions in the type-IIA string. In particular, we show that some of the $SL(2, \mathbb{R})$ duality of the type-IIA theory compactified to $d = 9$ has a natural interpretation in eleven dimensions: an $O(2)$ subgroup of $SL(2, \mathbb{R})$ can be interpreted as eleven-dimensional Lorentz transformations. The relevance of eleven-dimensional supergravity to the type-IIA string has been discussed independently by Witten [9].

The nine-dimensional theory is also expected to have an $O(1,1)$ or $SO(1,1)$ symmetry which is related to T-duality. Indeed, when truncated to the NS-NS sector, the nine-dimensional theory indeed has an $O(1,1) = SO(1,1) \times \mathbb{Z}_2$ symmetry which has a \mathbb{Z}_2 subgroup that corresponds to the expected $R \rightarrow \alpha'/R$ T-duality. However, only an $SO(1,1)$ subgroup extends to a symmetry of the full $N = 2$ theory in nine dimensions, while the \mathbb{Z}_2 " $R \rightarrow \alpha'/R$ " duality does not correspond to any such symmetry of the $d = 9$ supergravity. One of our aims is to elucidate the extension of this \mathbb{Z}_2 symmetry to the type-II theory and show how it can be understood in terms of supergravity theories.

We shall investigate the extent to which these type-II dualities can be interpreted as symmetries of $d = 10$ theories on backgrounds with one isometry and of $d = 11$ theories on backgrounds with two isometries. As has already been mentioned, the T-duality

gives a solution-generating transformation which takes type-IIA to type-IIB, and vice versa. As the type-IIB theory has no known eleven-dimensional origin, we will only be able to lift the nine-dimensional dualities to solution-generating transformations of $d = 11$ supergravity on backgrounds with two isometries for a special restricted class of backgrounds satisfying certain geometric and algebraic conditions. Eleven-dimensional supergravity is the low-energy limit of the eleven-dimensional supermembrane [13]; a search for supermembrane duality symmetries was undertaken in the context of a (three-dimensional) sigma-model description of supermembranes in Refs. [4,14]. We are able, in addition, to explicitly construct another set of solution-generating transformations that acts only inside each of the type-II strings on backgrounds with one isometry, behaves as a strong-weak coupling duality and is therefore part of $SL(2, \mathbb{R})$.

As an illustration of how our results can be applied to generate new solutions to the string equations of motion, we will consider in this paper the Supersymmetric String Wave (SSW) solution of Ref. [15] which is a solution of the heterotic string but also solves the type-II string equations of motion. Under a type-I T-duality transformation the SSW solution generates the Generalized Fundamental String Solution of Ref. [16] which is a generalization of the fundamental string solution [17]. We will show how the application of both type-II S- and T-dualities as well as combinations thereof generate new solutions of the type-IIA and type-IIB equations of motion. We lift the SSW solution to a solution of the eleven-dimensional theory. This gives a generalization of the eleven-dimensional pp-wave solution constructed in [18]. Applying a $d = 11$ type-I T-duality transformation generates an eleven-dimensional Generalized Fundamental Membrane (GFM) solution which is a generalization of the fundamental membrane solution of Ref. [19].

In exploring these symmetries, we work out some of the details of supergravity theories in $d = 9, 10$ that have not appeared in the literature before. We give the bosonic part of the $N = 2$, $d = 9$ action, which has not been written down explicitly before. We also write the type-IIA supergravity action for the stringy metric and find that, whereas the NS-NS fields appear with a coupling to the dilaton $\hat{\phi}$ through an overall factor of $e^{-2\hat{\phi}}$, as expected, the RR fields appear without any dilaton coupling. This follows from the fact that, on compactifying to four dimensions for example, the RR fields are invariant under S-duality [3] and this implies that they cannot couple to the dilaton in a way that respects S-duality; this was noticed independently by Witten [9]².

The organization of this work is as follows. In Section 2 we first derive the action of type-IIA supergravity in the "string-frame" metric. This action describes the zero-slope limit ($\alpha' \rightarrow 0$) of the type-IIA superstring. We use here dimensional reduction from eleven dimensions. In order to derive the type-II T-duality rules we first reduce in the next section the type-IIA supergravity theory to nine dimensions and thus obtain the action of $N = 2$, $d = 9$ supergravity. Next, in Section 4 we present the equations of motion of type-IIB supergravity in the "string-frame" metric and discuss its reduction to nine dimensions. Using all this information we derive in Section 5 the explicit

² It was already known that the four-dimensional dilaton-axion field cannot couple to RR vectors [20]

form of the above-mentioned type-II \mathcal{S} - and T-duality rules. In Section 6 we use our results to derive a type-I T-duality symmetry of $N = 1, d = 11$ supergravity, the only duality symmetry that can be found in this framework. Finally, as an illustration of how our results can be applied to generate new solutions, we will apply in Section 7 the duality transformations constructed in this work to the SSW solution and generate new, “dual”, solutions of the type-IIA and type-IIB superstrings and of eleven-dimensional supergravity. Our conventions are explained in Appendix A and Appendix B contains some useful formulae giving the explicit relations between certain eleven- and nine-dimensional fields.

2. The type-IIA superstring

The zero-slope limit of the type-IIA superstring corresponds to $N = 2, d = 10$ non-chiral supergravity. In this section we describe how to obtain the (bosonic sector of) type-IIA supergravity in the “string-frame” metric by dimensional reduction of $N = 1, d = 11$ supergravity. This can be done by a straightforward application of standard techniques (see for instance Ref. [21]). We describe the dimensional reduction in some detail since in order to derive the type-II duality rules (see Section 5) we need to know the exact relation between the supergravity theories in different dimensions. In this paper we will describe supergravity theories in $d = 9, 10$ and 11 dimensions. It is helpful to use a notation that clearly distinguishes between the different dimensions; throughout this paper we will use double hats for eleven-dimensional objects, single hats for ten-dimensional objects and no hats for nine-dimensional objects.

We now proceed to describe the dimensional reduction of $N = 1, d = 11$ supergravity [22]. The bosonic fields of this theory are the elfbein and a three-form potential

$$\left\{ \hat{\hat{e}}_{\hat{\mu}}^{\hat{a}}, \hat{\hat{C}}_{\hat{\mu}\hat{\nu}\hat{\rho}} \right\}. \quad (1)$$

The field strength of the three-form is

$$\hat{\hat{G}} = \partial \hat{\hat{C}}, \quad (2)$$

and the action for these bosonic fields is³

$$\hat{\hat{S}} = \frac{1}{2} \int d^{11}x \sqrt{\hat{\hat{g}}} \left[-\hat{\hat{R}} + a_1 \hat{\hat{G}}^2 + a_2 \frac{1}{\sqrt{\hat{\hat{g}}}} \hat{\hat{e}} \hat{\hat{G}} \hat{\hat{G}} \hat{\hat{C}} \right]. \quad (3)$$

We use the index-free notation explained in Appendix A. The coefficients a_1 and a_2 are numerical constants which are defined up to redefinitions of $\hat{\hat{C}}$, which implies that only the following quotient can be fixed:

$$a_1^3/a_2^2 = 9(4!)^3/2. \quad (4)$$

³ For simplicity, we have set the fermions to zero. It is straightforward to include fermion fields in the following analysis.

The action above is invariant under general coordinate transformations and the following gauge transformations of the $\hat{\mathcal{C}}$ potential:

$$\delta \hat{\mathcal{C}} = \partial \hat{\chi}. \quad (5)$$

We assume that all fields are independent of the coordinate $y = x^{10}$ which we choose to correspond to a space-like direction ($\hat{\eta}_{yy} = -1$) and we rewrite the fields and action in a ten-dimensional form. The dimensional reduction of the metric gives rise to the ten-dimensional metric, a vector field and a scalar (the dilaton) while the dimensional reduction of the three-form potential gives rise to a ten-dimensional three-form and a two-form. We thus obtain the fields of the ten-dimensional type-IIA supergravity theory which are

$$\left\{ \hat{\mathcal{C}}_{\hat{\mu}\hat{\nu}\hat{\rho}}, \hat{g}_{\hat{\mu}\hat{\nu}}, \hat{B}_{\hat{\mu}\hat{\nu}}^{(1)}, \hat{A}_{\hat{\mu}}^{(1)}, \hat{\phi} \right\}. \quad (6)$$

The eleven-dimensional fields can be expressed in terms of the ten-dimensional ones as follows

$$\begin{aligned} \hat{g}_{\hat{\mu}\hat{\nu}} &= e^{-\frac{2}{3}\hat{\phi}} \hat{g}_{\hat{\mu}\hat{\nu}} - e^{\frac{4}{3}\hat{\phi}} \hat{A}_{\hat{\mu}}^{(1)} \hat{A}_{\hat{\nu}}^{(1)}, & \hat{\mathcal{C}}_{\hat{\mu}\hat{\nu}\hat{\rho}} &= \hat{\mathcal{C}}_{\hat{\mu}\hat{\nu}\hat{\rho}}, \\ \hat{g}_{\hat{\mu}\underline{y}} &= -e^{\frac{4}{3}\hat{\phi}} \hat{A}_{\hat{\mu}}^{(1)}, & \hat{\mathcal{C}}_{\hat{\mu}\hat{\nu}\underline{y}} &= \frac{2}{3} \hat{B}_{\hat{\mu}\hat{\nu}}^{(1)}, \\ \hat{g}_{\underline{y}\underline{y}} &= -e^{\frac{4}{3}\hat{\phi}}. \end{aligned} \quad (7)$$

For the vielbeins we have

$$\left(\hat{e}_{\hat{\mu}}^{\hat{a}} \right) = \begin{pmatrix} e^{-\frac{1}{3}\hat{\phi}} \hat{e}_{\hat{\mu}}^{\hat{a}} & e^{\frac{2}{3}\hat{\phi}} \hat{A}_{\hat{\mu}}^{(1)} \\ 0 & e^{\frac{4}{3}\hat{\phi}} \end{pmatrix}, \quad \left(\hat{e}_{\hat{a}}^{\hat{\mu}} \right) = \begin{pmatrix} e^{\frac{1}{3}\hat{\phi}} \hat{e}_{\hat{a}}^{\hat{\mu}} & -e^{\frac{1}{3}\hat{\phi}} \hat{A}_{\hat{a}}^{(1)} \\ 0 & e^{-\frac{2}{3}\hat{\phi}} \end{pmatrix}. \quad (8)$$

Conversely, the ten-dimensional fields can be expressed in terms of the eleven-dimensional ones via

$$\begin{aligned} \hat{g}_{\hat{\mu}\hat{\nu}} &= \left(-\hat{g}_{\underline{y}\underline{y}} \right)^{\frac{1}{2}} \left(\hat{g}_{\hat{\mu}\hat{\nu}} - \hat{g}_{\hat{\mu}\underline{y}} \hat{g}_{\hat{\nu}\underline{y}} / \hat{g}_{\underline{y}\underline{y}} \right), & \hat{\mathcal{C}}_{\hat{\mu}\hat{\nu}\hat{\rho}} &= \hat{\mathcal{C}}_{\hat{\mu}\hat{\nu}\hat{\rho}}, \\ \hat{A}_{\hat{\mu}}^{(1)} &= \hat{g}_{\hat{\mu}\underline{y}} / \hat{g}_{\underline{y}\underline{y}}, & \hat{B}_{\hat{\mu}\hat{\nu}}^{(1)} &= \frac{3}{2} \hat{\mathcal{C}}_{\hat{\mu}\hat{\nu}\underline{y}}, \\ \hat{\phi} &= \frac{3}{4} \log \left(-\hat{g}_{\underline{y}\underline{y}} \right). \end{aligned} \quad (9)$$

The ten-dimensional fields have been defined in this way because, as we will see, (i) their gauge transformations are natural (no scalars are involved) and of a standard form (see below) and (ii) if we truncate the theory by setting $\hat{\mathcal{C}} = \hat{A}^{(1)} = 0$ we recover the bosonic action of $N = 1$, $d = 10$ (type-I) supergravity written with the usual conventions in the “string-frame” metric.

We now consider the reduction of the action (3) in more detail. We first consider the Ricci scalar term. To reduce this term we use (a slight generalization of) Palatini’s identity⁴:

⁴ Since the identity is valid in arbitrary d dimensions we do not use any hats here.

$$\begin{aligned} & \int d^d x \sqrt{|g|} e^{-2\phi} [-R] \\ &= \int d^d x \sqrt{|g|} e^{-2\phi} [\omega_b{}^{ba} \omega_c{}^c{}_a + \omega_a{}^{bc} \omega_{bc}{}^a + 4\omega_b{}^{ba} (\partial_a \phi)] . \end{aligned} \quad (10)$$

The non-vanishing components of the spin connection are

$$\begin{aligned} \hat{\omega}_{y\hat{a}y} &= -\frac{2}{3} e^{\frac{1}{3}\hat{\phi}} \partial_{\hat{a}} \hat{\phi} , & \hat{\omega}_{y\hat{a}\hat{b}} &= -\frac{1}{2} e^{\frac{4}{3}\hat{\phi}} \hat{F}_{\hat{a}\hat{b}}^{(1)} , \\ \hat{\omega}_{\hat{a}\hat{b}y} &= \frac{1}{2} e^{\frac{4}{3}\hat{\phi}} \hat{F}_{\hat{a}\hat{b}}^{(1)} , & \hat{\omega}_{\hat{a}\hat{b}\hat{c}} &= e^{\frac{1}{3}\hat{\phi}} \left(\hat{\omega}_{\hat{a}\hat{b}\hat{c}} + \frac{2}{3} \delta_{\hat{a}[\hat{b}} \partial_{\hat{c}]} \hat{\phi} \right) , \end{aligned} \quad (11)$$

where

$$\hat{F}^{(1)} = 2\partial \hat{A}^{(1)} \quad (12)$$

is the field strength of the ten-dimensional vector field $\hat{A}_{\hat{\mu}}^{(1)}$. Ignoring the integration over y and using

$$\sqrt{\hat{g}} = \sqrt{-\hat{g}} e^{-\frac{8}{3}\hat{\phi}} , \quad (13)$$

plus Palatini's identity (10) for $d = 11$ and $\phi = 0$ we find

$$\begin{aligned} & \frac{1}{2} \int d^{11} x \sqrt{\hat{g}} [-\hat{R}] \\ &= \frac{1}{2} \int d^{10} x \sqrt{-\hat{g}} \left\{ e^{-2\hat{\phi}} \left[\left(\hat{\omega}_{\hat{b}}{}^{\hat{b}\hat{a}} + 2\partial^{\hat{a}} \hat{\phi} \right)^2 + \hat{\omega}_{\hat{a}}{}^{\hat{b}\hat{c}} \hat{\omega}_{\hat{b}\hat{c}}{}^{\hat{a}} \right] + \frac{1}{4} (\hat{F}^{(1)})^2 \right\} . \end{aligned} \quad (14)$$

Finally, using Palatini's identity (10) again, but now for $d = 10$ and $\phi = \hat{\phi}$, we get for the Ricci-scalar term:

$$\begin{aligned} & \frac{1}{2} \int d^{11} x \sqrt{\hat{g}} [-\hat{R}] \\ &= \frac{1}{2} \int d^{10} x \sqrt{-\hat{g}} \left\{ e^{-2\hat{\phi}} \left[-\hat{R} + 4 \left(\partial \hat{\phi} \right)^2 \right] + \frac{1}{4} (\hat{F}^{(1)})^2 \right\} . \end{aligned} \quad (15)$$

We next reduce the &term in Eq. (3). Usually we identify field strengths in eleven and ten dimensions with flat indices, but in this case we also have to take into account the scaling of the ten-dimensional metric, and therefore we define

$$\hat{G}_{\hat{a}\hat{b}\hat{c}\hat{d}} = e^{-\frac{4}{3}\hat{\phi}} \hat{\hat{G}}_{\hat{a}\hat{b}\hat{c}\hat{d}} , \quad (16)$$

which leads to

$$\hat{G} = \partial \hat{C} - 2\hat{H}^{(1)} \hat{A}^{(1)} \quad (17)$$

where $\hat{H}^{(1)}$ is the field strength of the two-form $\hat{B}^{(1)}$

$$\hat{H}^{(1)} = \partial \hat{B}^{(1)} . \quad (18)$$

Observe that, in spite of the fact that there is a vector field present, the two-form field strength does not contain any Chern-Simons term.

The remaining components of $\hat{\hat{G}}$ are given by

$$\hat{\hat{G}}_{\hat{a}\hat{b}\hat{c}y} = \frac{1}{2} e^{\frac{1}{3}\hat{\phi}} \hat{H}_{\hat{a}\hat{b}\hat{c}}^{(1)}, \quad (19)$$

and the contribution of the $\hat{\hat{G}}$ -term to the ten-dimensional action becomes

$$\frac{1}{2} \int d^{11}x \sqrt{\hat{\hat{g}}} a_1 (\hat{\hat{G}})^2 = \frac{1}{2} \int d^{10}x \sqrt{-\hat{g}} \left[-a_1 e^{-2\hat{\phi}} (\hat{H}^{(1)})^2 + a_1 \hat{\hat{G}}^2 \right]. \quad (20)$$

Finally, taking into account

$$\hat{\hat{\epsilon}}^{\mu_0 \dots \mu_9 y} = \hat{\epsilon}^{\mu_0 \dots \mu_9}, \quad (21)$$

the third term in the $d = 11$ action (3) (all terms with curved indices) gives

$$\hat{\hat{\epsilon}}^{\hat{\hat{G}}\hat{\hat{G}}\hat{\hat{G}}} = 2\hat{\epsilon}\partial\hat{C}\partial\hat{C}\hat{B}^{(1)} - 4\hat{\epsilon}\partial\hat{C}\partial\hat{B}^{(1)}\hat{C}, \quad (22)$$

and integrating by parts we get

$$\frac{1}{2} \int d^{11}x a_2 \hat{\hat{\epsilon}}^{\hat{\hat{G}}\hat{\hat{G}}\hat{\hat{G}}} = \frac{1}{2} \int d^{10}x \left[6a_2 \hat{\epsilon}\partial\hat{C}\partial\hat{C}\hat{B}^{(1)} \right]. \quad (23)$$

Collecting all our results and setting the constants at $= 3/4, a_2 = 1/384$ we find that the bosonic part of the type-IIA supergravity action in ten dimensions in the “string-frame” metric is given by⁵

$$\begin{aligned} \hat{S} = & \frac{1}{2} \int d^{10}x \sqrt{-\hat{g}} \left\{ e^{-2\hat{\phi}} \left[-\hat{R} + 4 \left(\partial\hat{\phi} \right)^2 - \frac{3}{4} \left(\hat{H}^{(1)} \right)^2 \right] \right. \\ & \left. + \frac{1}{4} \left(\hat{F}^{(1)} \right)^2 + \frac{3}{4} \hat{G}^2 + \frac{1}{64} \frac{\hat{\epsilon}}{\sqrt{-\hat{g}}} \partial\hat{C}\partial\hat{C}\hat{B}^{(1)} \right\}. \end{aligned} \quad (24)$$

The dilaton dependence here is at first sight rather surprising. The first line of Eq. (24) describes the fields from the NS-NS sector and is the same as the bosonic part of the type-1 supergravity action, and in particular has the expected dilaton dependence. The second line of Eq. (24) involves the fields from the RR sector – it vanishes in the truncation from type-IIA to type-1 supergravity:

$$\hat{C} = \hat{A}^{(1)} = 0. \quad (25)$$

– and is independent of the dilaton (as noted independently by Witten [24,9]). The absence of dilaton coupling to the RR fields reflects the fact that in four dimensions the RR fields are invariant under S-duality and that the RR charges are carried by solitons, not fundamental string modes [3].

Usually, the scalar sector in supergravity theories parametrizes a coset. In this case, there is only one scalar and the corresponding coset is trivially given by $SO(1,1)/1$ [25]. Nevertheless, it leads to a global $SO(1,1)$ invariance (with parameter α) which describes the coupling of the dilaton. Defining the scale weight w of a field A by $A \rightarrow$

⁵ The type-IIA action in the “Einstein-frame” metric has been given in [23].

Table 1

This table gives the weights w of the fields of type-IIA supergravity under the global $SO(1,1)$ symmetry

Field	Weight w	Field	Weight w
$e^{\hat{\phi}}$	1	$\hat{A}^{(1)}$	-3/4
\hat{g}	1/2	\hat{C}	-1/4
$\hat{B}^{(1)}$	1/2		

$e^{w\alpha}A$, the transformation rules under $SO(1,1)$ are specified by the weights in Table 1. This symmetry is the S-duality symmetry of the ten-dimensional type-IIA theory [3] and so we see that the ten-dimensional RR fields do transform under S-duality.

It is instructive to check the gauge invariance of the action (24). In eleven dimensions we have reparametrization invariance and the gauge symmetry (5). From the ten-dimensional point of view, only the reparametrization invariance of the eleven-dimensional theory in the direction parametrized by the coordinate y is relevant:

$$\delta y = -A^{(1)}(x), \quad (26)$$

where $A^{(1)}(x)$ does not depend on y . We only consider the infinitesimal form of the gauge transformations. Under Eq. (26) the eleven-dimensional fields transform as follows:

$$\begin{aligned} \delta \hat{g}_{\hat{\mu}\hat{\nu}} &= 2\hat{g}_{y(\hat{\mu}}\partial_{\hat{\nu})}A^{(1)}, & \delta \hat{g}_{\hat{\mu}y} &= \hat{g}_{yy}\partial_{\hat{\mu}}A^{(1)}, \\ \delta \hat{\hat{C}}_{\hat{\mu}\hat{\nu}\hat{\rho}} &= 3\hat{\hat{C}}_{y[\hat{\mu}\hat{\nu}}\partial_{\hat{\rho}]}A^{(1)}. \end{aligned} \quad (27)$$

These transformations and the gauge transformations of the three-form potential (5) reduce to the following transformations of the fields of the ten-dimensional theory

$$\begin{aligned} \delta \hat{A}^{(1)} &= \partial A^{(1)} & \delta \hat{B}^{(1)} &= \partial \hat{\eta}^{(1)}, \\ \delta \hat{C} &= \partial \hat{\chi} + 2\hat{B}^{(1)}\partial A^{(1)}, \end{aligned} \quad (28)$$

where the parameters of the ten-dimensional gauge transformations are related to the eleven-dimensional parameter $\hat{\chi}$ by

$$\hat{\chi}_{\hat{\mu}\hat{\nu}} = \hat{\chi}_{\hat{\mu}\hat{\nu}}, \quad \hat{\eta}_{\hat{\mu}}^{(1)} = \hat{\chi}_{\hat{\mu}y}. \quad (29)$$

It is easy to check that the ten-dimensional action (24) is invariant under the above gauge transformations.

3. Reduction to nine dimensions

We shall now compactify the type-IIA theory to $N = 2$ supergravity in nine dimensions to facilitate the derivation of the type-II duality rules. This is in accordance with the interpretation of duality as the non-compact symmetry of a compactified supergravity theory [5,3].

To reduce the first line of the type-IIA action given in Eq. (24) we can use the results for the heterotic string which have been given elsewhere (see e.g. Ref. [26]). Since we change notation slightly with respect to Ref. [26] we summarize some relevant formulae here. First we parametrize the zehnbein as follows:

$$(\hat{e}_{\hat{\mu}}^{\hat{a}}) = \begin{pmatrix} e_{\mu}^a & k A_{\mu}^{(2)} \\ 0 & k \end{pmatrix}, \quad (\hat{e}_{\hat{a}}^{\hat{\mu}}) = \begin{pmatrix} e_a^{\mu} & -A_a^{(2)} \\ 0 & k^{-1} \end{pmatrix}, \quad (30)$$

where

$$k = |\hat{k}_{\hat{\mu}} \hat{k}^{\hat{\mu}}|^{\frac{1}{2}}, \quad (31)$$

and $A_a^{(2)} = e_a^{\mu} A_{\mu}^{(2)}$. Here $\hat{k}_{\hat{\mu}}$ is a Killing vector such that

$$\hat{k}^{\hat{\mu}} \partial_{\hat{\mu}} = a, \quad (32)$$

This time we assume that all fields are independent of the coordinate $x = x^9$ which we choose to be a space-like direction ($\hat{\eta}_{xx} = -1$). Note that $\hat{k}_{\hat{\mu}} \hat{k}^{\hat{\mu}} = \hat{g}_{xx} = -k^2$.

Using the above zehnbeins, the ten-dimensional fields $\{\hat{g}_{\hat{\mu}\hat{\nu}}, \hat{B}_{\hat{\mu}\hat{\nu}}^{(1)}, \hat{\phi}\}$ decompose as follows:

$$\begin{aligned} \hat{g}_{\mu\nu} &= g_{\mu\nu} - k^2 A_{\mu}^{(2)} A_{\nu}^{(2)}, & \hat{B}_{\mu\nu}^{(1)} &= B_{\mu\nu}^{(1)} + A_{[\mu}^{(2)} B_{\nu]}, \\ \hat{g}_{xx} &= -k^2, & \hat{B}_{x\mu}^{(1)} &= B_{\mu}, \\ \hat{g}_{x\mu} &= -k^2 A_{\mu}^{(2)}, & \hat{\phi} &= \phi + \frac{1}{2} \log k, \end{aligned} \quad (33)$$

where $\{g_{\mu\nu}, B_{\mu\nu}^{(1)}, B_{\mu}, \phi, A_{\mu}^{(2)}, k\}$ are the nine-dimensional fields. They are given in terms of the ten-dimensional fields by

$$\begin{aligned} g_{\mu\nu} &= \hat{g}_{\mu\nu} - \hat{g}_{x\mu} \hat{g}_{x\nu} / \hat{g}_{xx}, & B_{\mu} &= \hat{B}_{x\mu}^{(1)}, \\ B_{\mu\nu}^{(1)} &= \hat{B}_{\mu\nu}^{(1)} + \hat{g}_{x[\mu} \hat{B}_{\nu]}^{(1)} / \hat{g}_{xx}, & \phi &= \hat{\phi} - \frac{1}{4} \log(-\hat{g}_{xx}), \\ A_{\mu}^{(2)} &= \hat{g}_{x\mu} / \hat{g}_{xx}, & k &= (-\hat{g}_{xx})^{\frac{1}{2}}. \end{aligned} \quad (34)$$

Therefore, ignoring the integral over x , the first line in the ten-dimensional action (24) can be written as

$$\begin{aligned} & \frac{1}{2} \int d^{10}x \sqrt{-\hat{g}} e^{-2\hat{\phi}} \left[-\hat{R} + 4(\partial\hat{\phi})^2 - \frac{3}{4}(\hat{H}^{(1)})^2 \right] \\ &= \frac{1}{2} \int d^9x \sqrt{g} e^{-2\phi} \left[-R + 4(\partial\phi)^2 - \frac{3}{4}(H^{(1)})^2 \right. \\ & \quad \left. - (\partial \log k)^2 + \frac{1}{4}k^2 (F^{(2)})^2 + \frac{1}{4}k^{-2} F^2(B) \right], \end{aligned} \quad (35)$$

where

$$\begin{aligned} F^{(2)} &= 2\partial A^{(2)}, & F(B) &= 2\partial B, \\ H^{(1)} &= \partial B^{(1)} + A^{(2)} \partial B + B \partial A^{(2)}. \end{aligned} \quad (36)$$

We next reduce the first term in the second line of Eq. (24). The vector field $\hat{A}^{(1)}$ reduces to a scalar and a vector as follows:

$$\begin{aligned}\hat{A}_{\underline{x}}^{(1)} &= \ell, \\ \hat{A}_{\mu}^{(1)} &= A_{\mu}^{(1)} + \ell A_{\mu}^{(2)}.\end{aligned}\quad (37)$$

We thus find

$$\int d^{10}x \sqrt{-\hat{g}} \left[\frac{1}{2} (\hat{F}^{(1)})^2 \right] = \int d^9x \sqrt{g} \left[\frac{1}{4} k (F^{(1)} + \ell F^{(2)})^2 - \frac{1}{2} k^{-1} (\partial \ell)^2 \right]. \quad (38)$$

To reduce the \hat{G}^2 term in Eq. (24) we decompose the three-index tensor \hat{C} as follows:

$$\begin{aligned}\hat{C}_{\mu\nu\underline{x}} &= \frac{2}{3} \left(B_{\mu\nu}^{(2)} - A_{[\mu}^{(1)} B_{\nu]} \right), \\ \hat{C}_{\mu\nu\rho} &= C_{\mu\nu\rho}.\end{aligned}\quad (39)$$

For the sake of completeness we also give the expression of the nine-dimensional fields $c_{\mu\nu\rho}, B_{\mu\nu}^{(2)}, A_{\mu}^{(1)}, e$ in terms of the ten-dimensional ones:

$$\begin{aligned}c_{\mu\nu\rho} &= \hat{C}_{\mu\nu\rho}, & B_{\mu\nu}^{(2)} &= \frac{3}{2} \hat{C}_{\mu\nu\underline{x}} - \hat{A}_{[\mu}^{(1)} \hat{B}_{\nu]\underline{x}} + \hat{g}_{\underline{x}[\mu} \hat{B}_{\nu]\underline{x}} \hat{A}_{\underline{x}}^{(1)} / \hat{g}_{\underline{x}\underline{x}}, \\ e &= \hat{A}_{\underline{x}}^{(1)}, & A_{\mu}^{(1)} &= \hat{A}_{\mu}^{(1)} - \hat{A}_{\underline{x}}^{(1)} \hat{g}_{x\mu} / \hat{g}_{xx}.\end{aligned}\quad (40)$$

We find that

$$\hat{G}_{abcx} = \frac{1}{2} k^{-1} \left(H_{abc}^{(2)} - \ell H_{abc}^{(1)} \right), \quad (41)$$

with

$$H^{(2)} = \partial B^{(2)} - A^{(1)} \partial B - B \partial A^{(1)} \quad (42)$$

At this point it is convenient to make use of the global O(2) invariance of the $N=2$, $d=9$ supergravity theory explained in Section 5 (see also Appendix B) and to write the field strengths $H^{(1)}$ and $H^{(2)}$ as

$$H^{(i)} = \partial B^{(i)} + \epsilon^{ij} (A^{(j)} \partial B + B \partial A^{(j)}), \quad i=1,2, \quad \epsilon^{12} = -\epsilon^{21} = +1. \quad (43)$$

Similarly we find that

$$\hat{G}_{abcd} = G_{abcd} \quad (44)$$

with

$$G = \partial C + 2A^{(i)} \partial B^{(i)} - 2\epsilon^{ij} B A^{(i)} \partial A^{(j)}. \quad (45)$$

We thus find that the \hat{G}^2 term in Eq. (24) is given by

$$\int d^{10}x \sqrt{-\hat{g}} \left[\frac{3}{4} \hat{G}^2 \right] = \int d^9x \sqrt{g} \left[\frac{3}{4} k G^2 - \frac{3}{4} k^{-1} (H^{(2)} - \ell H^{(1)})^2 \right]. \quad (46)$$

Finally, we reduce the $\hat{\epsilon} \partial \hat{C} \partial \hat{C} \hat{B}^{(1)}$ term in Eq. (24). A straightforward application of the previous formulae gives

$$\begin{aligned} \epsilon \partial \hat{C} \partial \hat{C} \hat{B}^{(1)} = \epsilon \{ & -2\partial C \partial CB - 2\partial C \partial B^{(i)} B^{(j)} \epsilon^{ij} \\ & -4\partial C A^{(i)} \partial B^{(i)} B + 2\partial C A^{(i)} A^{(j)} \partial BB \epsilon^{ij} \} . \end{aligned} \quad (47)$$

In summary, the fields of the $N = 2$, $d = 9$ supergravity theory are given by

$$\{g_{\mu\nu}, C_{\mu\nu\rho}, B_{\mu\nu}^{(i)}, A_{\mu}^{(i)}, B_{\mu}, \phi, k, \ell\} . \quad (48)$$

The action for these fields is given by

$$\begin{aligned} S = & \frac{1}{2} \int d^9 x \sqrt{g} \left\{ e^{-2\phi} \left[-R + 4(\partial\phi)^2 - \frac{3}{4}(H^{(1)})^2 \right. \right. \\ & \left. \left. - (\partial \log k)^2 + \frac{1}{4}k^2 (F^{(2)})^2 + \frac{1}{4}k^{-2} F^2(B) \right] \right. \\ & + \frac{1}{4}k (F^{(1)} + \ell F^{(2)})^2 - \frac{1}{2}k^{-1} (\partial\ell)^2 + \frac{3}{4}k G^2 - 34k^{-1} (H^{(2)} - \ell H^{(1)})^2 \\ & \left. - \frac{1}{32} \frac{1}{\sqrt{g}} \epsilon (\partial C \partial CB + \partial C \partial B^{(i)} B^{(j)} \epsilon^{ij} + 2\partial C A^{(i)} \partial B^{(i)} B - \partial C A^{(i)} A^{(j)} \partial BB \epsilon^{ij}) \right\} . \end{aligned} \quad (49)$$

In Ref. [27], it was suggested that the $N = 2, d = 9$ supergravity action should have a global $GL(2, \mathbb{R}) = SL(2, \mathbb{R}) \times SO(1, 1)$ invariance⁶. However, on physical grounds, one would expect a symmetry group containing at least the S-duality group $SL(2, \mathbb{R})$ and the T-duality group $O(1, 1) = SO(1, 1) \times \mathbb{Z}_2$, that is, $GL(2, \mathbb{R}) \times \mathbb{Z}_2$. As we shall see in Section 5, the invariance is indeed $GL(2, \mathbb{R})$ and the “missing” \mathbb{Z}_2 invariance will be the main theme of Section 5; it is related to the T-duality of the type-II theory.

It is instructive to consider the gauge invariances of this action. In ten dimensions we have the **reparametrizations** in the x -direction with a parameter $\Lambda^{(2)}(x^\mu)$ independent of x and the gauge transformations (28). After dimensional reduction they become the following symmetries of the nine-dimensional theory:

$$\begin{aligned} \delta A^{(i)} &= \partial \Lambda^{(i)} , & \delta B &= \partial \Lambda , \\ \delta B^{(i)} &= \partial \eta^{(i)} - \epsilon^{ij} (B \partial \Lambda^{(j)} + A^{(j)} \partial \Lambda) , \\ \delta C &= \partial \chi + 2B^{(i)} \partial \Lambda^{(i)} + 2BA^{(i)} \partial \Lambda^{(j)} \epsilon^{ij} , \end{aligned} \quad (50)$$

where the parameters of the nine-dimensional gauge transformations are related to the ten-dimensional parameters by

$$\begin{aligned} A &= -\frac{1}{2} \hat{\eta}_{\pm}^{(1)} , & \chi_{\mu\nu} &= \hat{\chi}_{\mu\nu} , \\ \eta_{\mu}^{(1)} &= \hat{\eta}_{\mu}^{(1)} , & \eta_{\mu}^{(2)} &= \hat{\chi}_{\mu\pm} . \end{aligned} \quad (51)$$

It is straightforward to check that the nine-dimensional action (49) is invariant under the above gauge transformations.

⁶ Any matrix of the group $GL(2, \mathbb{R})$ can be uniquely written as the product of an $SL(2, \mathbb{R})$ matrix, a real positive number and $+1$ or -1 . This gives the decomposition $GL(2, \mathbb{R}) = SL(2, \mathbb{R}) \times \mathbb{R}^+ \times \mathbb{Z}_2$. (The multiplicative group of the real positive number \mathbb{R}^+ is isomorphic to the additive group \mathbb{R} .) Finally, $SO(1, 1) = \mathbb{R}^+ \times \mathbb{Z}_2$.

4. The type-IIB superstring

We shall also need to consider the low-energy limit of the type-IIB superstring for our discussion duality. The zero-slope limit of the type-IIB superstring is given by $N = 2, d = 10$ chiral supergravity [28,29]. This theory contains a metric, a complex antisymmetric tensor, a complex scalar and a four-index antisymmetric tensor gauge field. The complex scalar parametrizes the coset $SU(1,1)/U(1)$. In order to distinguish between the type-IIA and type-IIB fields, we denote the type-IIB fields as follows:

$$\left\{ \hat{D}_{\hat{\mu}\hat{\nu}\hat{\lambda}\hat{\rho}}, \hat{h}_{\hat{\mu}\hat{\nu}}, \hat{B}_{\hat{\mu}\hat{\nu}}, \hat{\Phi} \right\}, \quad (52)$$

where $\hat{h}_{\hat{\mu}\hat{\nu}}$ is the ‘‘Einstein-frame’’ metric. We will start in the ‘‘Einstein-frame’’ and then switch to the ‘‘string-frame’’ metric once we have correctly identified the type-IIB dilaton field.

The field equations of the type-IIB theory cannot be derived from a covariant action. The type-IIB field equations of Ref. [28] are given (in our notation and conventions) by

$$\begin{aligned} \hat{R}_{\hat{\mu}\hat{\nu}}(\hat{h}) &= -2\hat{P}_{(\hat{\mu}}\hat{P}_{\hat{\nu})}^* - \frac{25}{6}\hat{F}(\hat{D})_{\hat{\lambda}_1\cdots\hat{\lambda}_4\hat{\mu}}\hat{F}(\hat{D})^{\hat{\lambda}_1\cdots\hat{\lambda}_4}{}_{\hat{\nu}} \\ &\quad - \frac{9}{4}\hat{G}_{(\hat{\mu}}^{\hat{\lambda}\hat{\rho}}\hat{G}_{\hat{\nu})\hat{\lambda}\hat{\rho}}^* + \frac{3}{16}\hat{h}_{\hat{\mu}\hat{\nu}}\hat{G}\hat{G}^*, \\ \nabla^{\hat{\lambda}}\hat{G}_{\hat{\mu}\hat{\nu}\hat{\lambda}} &= \frac{1}{2}\hat{Q}^{\hat{\lambda}}\hat{G}_{\hat{\mu}\hat{\nu}\hat{\lambda}} + \hat{P}^{\hat{\lambda}}\hat{G}_{\hat{\mu}\hat{\nu}\hat{\lambda}}^* - \frac{10}{3}i\hat{F}(\hat{D})_{\hat{\mu}\hat{\nu}\hat{\lambda}\hat{\rho}\hat{\sigma}}\hat{G}^{\hat{\lambda}\hat{\rho}\hat{\sigma}}, \\ \nabla^{\hat{\mu}}\hat{P}_{\hat{\mu}} &= \hat{Q}^{\hat{\lambda}}\hat{P}_{\hat{\lambda}} - \frac{3}{8}\hat{G}^2, \\ \hat{F}(\hat{D}) &= \tilde{F}(\hat{D}). \end{aligned} \quad (53)$$

We have used here the following definitions:

$$\begin{aligned} \hat{G} &= \frac{\hat{\mathcal{H}} - \hat{\Phi}\hat{\mathcal{H}}^*}{(1 - \hat{\Phi}^*\hat{\Phi})^{1/2}}, \quad \text{with} \quad \mathcal{H} = \partial\mathcal{B}, \\ \hat{F}(\hat{D}) &= \partial\hat{D} - \frac{3}{8i}\left(\hat{B}\partial\hat{B}^* - \hat{B}^*\partial\hat{B}\right), \\ \hat{P} &= \frac{\partial\hat{\Phi}}{1 - \hat{\Phi}^*\hat{\Phi}}, \quad \hat{Q} = \frac{\hat{\Phi}\overleftrightarrow{\partial}\hat{\Phi}^*}{1 - \hat{\Phi}^*\hat{\Phi}}. \end{aligned} \quad (54)$$

The theory is invariant under $d = 10$ general coordinate transformations and under the following tensor gauge transformations:

$$\begin{aligned} \delta\hat{B} &= \partial\hat{\Sigma}, \\ \delta\hat{D} &= \partial\hat{\rho} + \frac{3}{8i}\left(\partial\hat{\Sigma}\hat{B}^* - \partial\hat{\Sigma}^*\hat{B}\right). \end{aligned} \quad (55)$$

It is known that the dimensional reduction of $d = 10$ type-IIA and IIB supergravity leads to the same $N = 2, d = 9$ supergravity theory. Our task is to make the correct identifications between the dimensionally reduced type-IIB fields and the fields of

$N=2$, $d=9$ supergravity as found in the previous section. It is convenient to start by rewriting the theory using the “string-frame” metric $\hat{J}_{\hat{\mu}\hat{\nu}}$, but before we have to identify the type-IIB dilaton. This is easier to do in the $SL(2, \mathbb{R})$ version of the theory. Accordingly, we first define the complex scalar field $\hat{\lambda} = \hat{\ell} + ie^{-\hat{\phi}}$ by

$$-i\hat{\lambda} = \frac{1 - \hat{\phi}}{1 + \hat{\phi}}, \quad (56)$$

which gives

$$\frac{\partial_{\hat{\mu}} \hat{\phi} \partial^{\hat{\mu}} \hat{\phi}^*}{(1 - \hat{\phi} \hat{\phi}^*)^2} = \frac{1}{4} \frac{\partial_{\hat{\mu}} \hat{\lambda} \partial^{\hat{\mu}} \hat{\lambda}^*}{(\text{Im } \hat{\lambda})^2}, \quad (57)$$

so $\hat{\lambda}$ parametrizes an $SL(2, \mathbb{R})$ coset. We next define the “string-frame” metric $\hat{J}_{\hat{\mu}\hat{\nu}}$ by

$$\hat{J}_{\hat{\mu}\hat{\nu}} = e^{\frac{1}{2}\hat{\phi}} \hat{h}_{\hat{\mu}\hat{\nu}}. \quad (58)$$

This definition implies that $\hat{\phi}$ is the type-IIB dilaton and will be justified below. We next consider the complex antisymmetric tensor $\hat{\mathcal{B}}$. To make contact with the “real” $O(2)$ notation of the previous section we write

$$\hat{\mathcal{B}} = \hat{\mathcal{B}}^{(1)} + i\hat{\mathcal{B}}^{(2)} \quad \hat{\Sigma} = \hat{\Sigma}^{(1)} + i\hat{\Sigma}^{(2)} \quad (59)$$

Using this notation the field strengths of the $\hat{\mathcal{B}}$ gauge fields and their gauge transformations can be written as:

$$\begin{aligned} \hat{\mathcal{H}}^{(i)} &= \partial \hat{\mathcal{B}}^{(i)}, & \delta \hat{\mathcal{B}}^{(i)} &= \partial \hat{\Sigma}^{(i)}, \\ \hat{F}(\hat{D}) &= \partial \hat{D} + \frac{3}{4} \epsilon^{ij} \hat{\mathcal{B}}^{(i)} \partial \hat{\mathcal{B}}^{(j)}, & \delta \hat{D} &= \partial \hat{\rho} - \frac{3}{4} \epsilon^{ij} \partial \hat{\Sigma}^{(i)} \hat{\mathcal{B}}^{(j)}. \end{aligned} \quad (60)$$

To explain why it is appropriate to identify the type-III3 dilaton with the $\hat{\phi}$ scalar field it is convenient to use the following trick. Although there is no action in ten dimensions giving rise to the full type-IIB field equations, it turns out that one can write down an action giving rise to the type-IIB field equations with $\hat{F}(\hat{D}) = 0$. This action is given by

$$\hat{S}_{\text{IIB}}^{\text{sugra}} = \frac{1}{2} \int d^{10}x \sqrt{-\hat{h}} \left[-\hat{R}(\hat{h}) - 2 \frac{\partial \hat{\phi} \partial \hat{\phi}^*}{(1 - \hat{\phi}^* \hat{\phi})^2} - \frac{3}{4} \hat{G}^* \hat{G} \right]. \quad (61)$$

If we now perform all the above changes in this action we get the following action in the “string-frame” metric:

$$\begin{aligned} \hat{S}_{\text{IIB}}^{\text{string}} &= \frac{1}{2} \int d^{10}x \sqrt{-\hat{J}} \left\{ e^{-2\hat{\phi}} \left[-\hat{R}(\hat{J}) + 4(\partial \hat{\phi})^2 - \frac{3}{4} (\hat{\mathcal{H}}^{(1)})^2 \right] \right. \\ &\quad \left. - \frac{1}{2} (\partial \hat{\ell})^2 - \frac{3}{4} (\hat{\mathcal{H}}^{(2)} - \hat{\ell} \hat{\mathcal{H}}^{(1)})^2 \right\}. \end{aligned} \quad (62)$$

It is easy to read from this action that the truncation $\hat{D} = \hat{\mathcal{B}}^{(2)} = \hat{\ell} = 0$ (which implies $\hat{F}(\hat{D}) = 0$, so it is consistent to use this action) gives the usual type-1 action. We see

that, as in the **type-IIA** case, the **type-IIB** RR fields do not appear multiplied by the string coupling constant (the **dilaton**).

The equations of motion for the full **type-IIB** theory written in terms of the “stringy” fields

$$\left\{ \hat{D}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}, \hat{J}_{\hat{\mu}\hat{\nu}}, \hat{B}_{\hat{\mu}\hat{\nu}}^{(i)}, \mathcal{L}, \hat{\phi} \right\} \quad (63)$$

are

$$\begin{aligned} \hat{R}_{\hat{\mu}\hat{\nu}}(\hat{J}) &= 4\partial_{\hat{\mu}}\hat{\phi}\partial_{\hat{\nu}}\hat{\phi} - \frac{9}{4}\hat{\mathcal{H}}_{(\hat{\mu}}^{(1)\hat{\lambda}\hat{\rho}}\hat{\mathcal{H}}_{\hat{\nu})\hat{\lambda}\hat{\rho}}^{(1)} + \frac{3}{16}\hat{J}_{\hat{\mu}\hat{\nu}}\left(\hat{\mathcal{H}}^{(1)}\right)^2 \\ &\quad + e^{2\hat{\phi}}\left[-\frac{1}{2}\partial_{\hat{\mu}}\hat{\ell}\partial_{\hat{\nu}}\hat{\ell} - \frac{9}{4}\left(\hat{\mathcal{H}}^{(2)} - \hat{\ell}\hat{\mathcal{H}}^{(1)}\right)_{(\hat{\mu}}^{\hat{\lambda}\hat{\rho}}\left(\hat{\mathcal{H}}^{(2)} - \hat{\ell}\hat{\mathcal{H}}^{(1)}\right)_{\hat{\nu})\hat{\lambda}\hat{\rho}}\right. \\ &\quad \left.+ \frac{3}{16}\hat{J}_{\hat{\mu}\hat{\nu}}\left(\hat{\mathcal{H}}^{(2)} - \hat{\ell}\hat{\mathcal{H}}^{(1)}\right)^2 - \frac{25}{6}\hat{F}(\hat{D})_{\hat{\lambda}_1\ldots\hat{\lambda}_4\hat{\mu}}\hat{F}(\hat{D})^{\hat{\lambda}_1\ldots\hat{\lambda}_4}_{\hat{\nu}}\right], \\ \nabla^2\hat{\phi} &= \frac{1}{4}\hat{R}(\hat{J}) + \frac{3}{16}\left(\hat{\mathcal{H}}^{(1)}\right)^2 + (\partial\hat{\phi})^2, \\ \nabla^2\hat{\ell} &= -\frac{3}{2}\hat{\mathcal{H}}^{(1)}\left(\hat{\mathcal{H}}^{(2)} - \hat{\ell}\hat{\mathcal{H}}^{(1)}\right), \\ \nabla^{\hat{\mu}}\left[\left(\hat{\ell}^2 + e^{-2\hat{\phi}}\right)\hat{\mathcal{H}}^{(1)} - \hat{\ell}\hat{\mathcal{H}}^{(2)}\right]_{\hat{\mu}\hat{\nu}\hat{\rho}} &= \frac{10}{3}\hat{F}(\hat{D})_{\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\lambda}\hat{\kappa}}\hat{\mathcal{H}}^{(2)\hat{\sigma}\hat{\lambda}\hat{\kappa}}, \\ \nabla^{\hat{\mu}}\left(\hat{\mathcal{H}}^{(2)} - \hat{\ell}\hat{\mathcal{H}}^{(1)}\right)_{\hat{\mu}\hat{\nu}\hat{\rho}} &= -\frac{10}{3}\hat{F}(\hat{D})_{\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\lambda}\hat{\kappa}}\hat{\mathcal{H}}^{(1)\hat{\sigma}\hat{\lambda}\hat{\kappa}}, \\ \hat{F}(\hat{D}) &= \tilde{F}(\hat{D}). \end{aligned} \quad (64)$$

In the second equation of (64) we can see that, although the RR fields do not couple directly to the dilaton, they couple indirectly to it through the metric.

This is going to be our starting point for the dimensional reduction to $d = 9$. First we want the dimensional reduction of $\hat{\mathcal{H}}^{(i)}$ to reproduce the nine-dimensional field strengths $H^{(i)}$ given in Eq. (43). We observe that $\hat{\mathcal{H}}^{(i)}$ contains no Chern-Simons term while $H^{(i)}$ does. This means that in the **type-IIB** reduction one of the vector fields present in the Chern-Simons part of $H^{(i)}$ must be identified with the vector field present in the parametrization of the type-IIB zehnbein. In the **type-IIA** reduction this vector field was called $A^{(2)}$ (see Eq. (30)). Note that the vector field $A^{(2)}$ is present in $H^{(2)}$ but not in $H^{(1)}$ so we cannot use the same parametrization⁷. We see that on the other hand the vector field B does occur in the Chern-Simons part of both $H^{(1)}$ and $H^{(2)}$. Therefore B must occur in the parametrization of the type-IIB zehnbein. At this point we realize that the NS-NS string part of the nine-dimensional action (i.e. the first two lines in Eq. (49)) are invariant under the \mathbb{Z}_2 transformation

$$\tilde{A}_{\mu}^{(2)} = B_{\mu}, \quad \tilde{B}_{\mu} = A_{\mu}^{(2)}, \quad \tilde{k} = k^{-1}. \quad (65)$$

⁷ The situation in the type-IIA reduction is different since there $B^{(2)}$ is related to \hat{C} whose field strength already contains a Chern-Simons term in ten dimensions.

5. Type-II S- and T-duality

In this section we shall find the type-II S - and T -duality rules described in the introduction. We start by exploring the non-compact symmetries of the type-II supergravity theory in nine dimensions and then seek their analogues in the “parent” theories in ten and (in the next section) eleven dimensions.

We start by considering the $SL(2, \mathbb{R})$ S -duality symmetry. The $SL(2, \mathbb{R})$ symmetry of the type-IIB theory in $d = 10$ gives rise to an $SL(2, \mathbb{R})$ symmetry of the $N = 2$ theory in $d = 9$. An $O(2)$ subgroup of this is a manifest symmetry of the action (49). Under $SL(2, \mathbb{R})$, $A_\mu^{(i)}$ and $B_{\mu\nu}^{(i)}$ are both doublets while $\lambda = \ell + ie^{-\phi}$ is a complex coordinate on $SL(2, \mathbb{R})/U(1)$ transforming by fractional linear transformations. The origin of this $SL(2, \mathbb{R})$ symmetry from the type-IIA theory is more subtle. An $SO(1, 1)$ subgroup which acts by shifting the dilaton arises from the $SO(1, 1)$ symmetry of the type-IIA theory in $d = 10$ discussed in Section 2. An $O(2)$ subgroup has a natural interpretation as Lorentz transformations of the **eleven-dimensional** supergravity in a background with two commuting isometries. We now discuss this $O(2)$ subgroup in more detail.

The eleven-dimensional theory is obviously invariant under the group $O(2) = SO(2) \times \mathbb{Z}_2$ of rotations and reflections in the xy plane, inducing an $O(2)$ invariance of the **nine-dimensional** theory. The infinitesimal form of the $SO(2)$ transformations of the scalars and vector fields is

$$\begin{aligned} \delta k &= \frac{1}{2} \theta k \ell, & \delta A^{(1)} &= -\theta A^{(2)}, \\ \delta e^\phi &= -\frac{7}{4} \theta \ell e^\phi, & \delta A^{(2)} &= \theta A^{(1)}, \\ \delta \ell &= \theta (\ell^2 + k e^{-2\phi}), & \delta B &= 0, \end{aligned} \quad (72)$$

and those of the remaining fields are

$$\begin{aligned} \delta B^{(1)} &= -\theta B^{(2)}, & \delta B^{(2)} &= \theta B^{(1)}, \\ \delta g_{\mu\nu} &= -\theta \ell g_{\mu\nu}, & \delta C &= 0, \end{aligned} \quad (73)$$

where θ is an infinitesimal constant parameter. On the other hand, the discrete \mathbb{Z}_2 transformations, corresponding to the reflection $y \rightarrow -y$, is given by

$$\begin{aligned} \ell' &= -\ell, & A^{(1)'} &= -A^{(1)} \\ B^{(1)'} &= -B^{(1)}, & B' &= -B, \end{aligned} \quad (74)$$

and the remaining fields are invariant. A particularly interesting $O(2)$ -rotated version of this \mathbb{Z}_2 transformation is given by an interchange of the coordinates x and y ⁸, under which the nine-dimensional scalars and vectors transform as follows:

$$\begin{aligned} k' &= k (\ell^2 + k e^{-2\phi})^{-\frac{1}{4}}, & A^{(1)'} &= A^{(2)}, \\ \ell' &= \ell (\ell^2 + k e^{-2\phi})^{-1}, & A^{(2)'} &= A^{(1)} \\ e^{\phi'} &= e^\phi (\ell^2 + k e^{-2\phi})^{\frac{7}{8}}, & B' &= -B, \end{aligned} \quad (75)$$

⁸ This transformation corresponds to a finite $O(2)$ rotation with parameter $\theta = -\pi/2$ followed by the reflection $y \rightarrow -y$.

and the remaining fields

$$\begin{aligned} B^{(1)'} &= B^{(2)}, & B^{(1)'} &= B^{(2)}, \\ g'_{\mu\nu} &= (\ell^2 + ke^{-2\phi})^{\frac{1}{2}} g_{\mu\nu}, & C' &= c. \end{aligned} \quad (76)$$

We now consider the ten-dimensional reformulation of these symmetries. The nine-dimensional $O(2)$ invariance Eqs. (72), (73), (74) corresponds to non-trivial dualities of both ten-dimensional type-II supergravity theories. As an example of this kind of duality we write down the ten-dimensional type-II transformations corresponding to the finite \mathbb{Z}_2 transformations given in Eqs. (75,76). We will provisionally call this \mathbb{Z}_2 transformation a type-II “xy-duality”. The explicit form of the type-IIA xy-duality rules is given by

$$\begin{aligned} \hat{\phi}' &= \hat{\phi} + \frac{3}{4} \log \hat{G}_{xx}, & \hat{A}_x^{(1)'} &= \hat{A}_x^{(1)} \hat{G}_{xx}^{-1}, \\ \hat{B}_{\mu\nu}^{(1)'} &= \frac{3}{2} \hat{C}_{\mu\nu x}, & \hat{A}_\mu^{(1)'} &= \left(\hat{A}_x^{(1)} \hat{A}_\mu^{(1)} - e^{-2\hat{\phi}} \hat{g}_{\mu x} \right) \hat{G}_{xx}^{-1}, \\ \hat{B}_{x\mu}^{(1)'} &= -\hat{B}_{x\mu}^{(1)}, & \hat{g}'_{xx} &= \hat{g}_{xx} \hat{G}_{xx}^{-\frac{1}{2}}, \\ \hat{g}'_{\mu x} &= 2 \hat{A}_{[\mu}^{(1)} \hat{g}_{x]} \hat{G}_{xx}^{-\frac{1}{2}}, & \hat{g}'_{\mu\nu} &= -e^{2\hat{\phi}} \hat{G}_{xx}^{-\frac{1}{2}} (\hat{G}_{xx} \hat{G}_{\mu\nu} - \hat{G}_{\mu x} \hat{G}_{\nu x}), \\ \hat{C}'_{\mu\nu\rho} &= \hat{C}_{\mu\nu\rho}, & \hat{C}'_{\mu\nu x} &= \frac{2}{3} \hat{B}_{\mu\nu}^{(1)}, \end{aligned} \quad (77)$$

where

$$\hat{G}_{\hat{\mu}\hat{\nu}} = \hat{A}_{\hat{\mu}}^{(1)} \hat{A}_{\hat{\nu}}^{(1)} - e^{-2\hat{\phi}} \hat{g}_{\hat{\mu}\hat{\nu}}. \quad (78)$$

Similarly, the type-IIB xy-duality transformations are given by

$$\begin{aligned} \hat{D}'_{\mu\nu\rho x} &= \hat{D}_{\mu\nu\rho x}, & \hat{J}'_{\mu\nu} &= |\hat{\lambda}| \hat{J}_{\mu\nu}, \\ \hat{J}'_{x\mu} &= -|\hat{\lambda}| \hat{J}_{x\mu}, & \hat{J}'_{xx} &= |\hat{\lambda}| \hat{J}_{xx}, \\ \hat{B}_{\mu\nu}^{(1)'} &= \hat{B}_{\mu\nu}^{(2)}, & \hat{B}_{\mu\nu}^{(2)'} &= \hat{B}_{\mu\nu}^{(1)}, \\ \hat{B}_{x\mu}^{(1)'} &= -\hat{B}_{x\mu}^{(2)}, & \hat{B}_{x\nu}^{(2)'} &= -\hat{B}_{x\nu}^{(1)}, \\ \hat{\ell}' &= |\hat{\lambda}|^{-2} \hat{\ell}, & \hat{\phi}' &= \hat{\phi} + 2 \log |\hat{\lambda}|, \end{aligned} \quad (79)$$

(recall that $\hat{\lambda} = \hat{\ell} + ie^{-\hat{\phi}}$).

Observe that the xy-dualities interchange (and mix) NS-NS fields with RR ones, and can be used to generate solutions with non-trivial RR fields from solutions of the NS-NS sector (which are also solutions of the heterotic string with no background gauge fields). In the type-IIB theory the xy-duality transformations Eqs. (79) is the S-duality transformation under which

$$\hat{\lambda}' = -1/X, \quad (80)$$

combined with other discrete symmetries of the theory.

In the case of the **type-IIA** theory, the xy-duality has its origin in the $O(2)$ symmetry of the eleven-dimensional theory restricted to backgrounds with two commuting isometries.

The type-IIA theory, when restricted to backgrounds with one isometry, has an $SL(2, \mathbb{R})$ S-duality invariance which includes the xy-duality Eqs. (77). Note that if we set $\hat{A}^{(1)} = 0$, $\hat{g}_{xx} = -1$ in Eqs. (77) (for simplicity) then the type-IIA xy-duality

transformation relates the strong- and weak-coupling regimes of the underlying type-IIA superstring theory:

$$\hat{\phi}' = -\frac{1}{2}\hat{\phi}. \quad (81)$$

Note also that Eqs. (77) and (79) are related by a type-II T-duality transformation as will be discussed below.

We now consider the construction of the type-II T-duality rules. It turns out that the derivation of these rules is rather subtle since the type-II T-transformations do not correspond to a non-compact symmetry of the nine-dimensional theory. As mentioned in the introduction, this is related to the fact that the type-II T duality maps one theory (the type-IIA superstring) onto another theory (the type-IIB superstring). Consider first the NS-NS truncation of the nine-dimensional theory, with the type-I action

$$S = \frac{1}{2} \int d^9x \sqrt{g} e^{-2\phi} \left[-R + 4(\partial\phi)^2 - \frac{3}{4}(H^{(1)})^2 - (\partial \log k)^2 + \frac{1}{4}k^2 (F^{(2)})^2 + \frac{1}{4}k^{-2}F^2(B) \right]. \quad (82)$$

This has an $O(1,1) = SO(1,1) \times \mathbb{Z}_2$ duality symmetry. The nine-dimensional \mathbb{Z}_2 transformation is given by

$$\tilde{A}_\mu^{(2)} = B_\mu, \quad \tilde{B}_\mu = A_\mu^{(2)}, \quad \tilde{k} = k^{-1}. \quad (83)$$

This is the standard T-duality transformation [10]. (Note that k is the modulus field for the compactifying circle, so that its expectation value corresponds to the radius R .) The continuous $SO(1,1)$ symmetry scales k and acts by

$$\tilde{k} = \Lambda k, \quad \tilde{B}_\mu = \Lambda B_\mu, \quad \tilde{A}_\mu^{(2)} = \Lambda^{-1} A_\mu^{(2)}. \quad (84)$$

This corresponds to a particular general coordinate transformation in $d = 10^9$.

The $SO(1,1)$ transformations extend to a symmetry of the full $d = 9$ type-II action (49) under which each field A scales with some weight w : $A \rightarrow \Lambda^w A$. The weights of the fields are given in Table 2.

However, the \mathbb{Z}_2 transformations (83) do not extend to any symmetry of the $d = 9$ action. Thus the T-duality transformations relating type-IIA backgrounds to type-IIB ones cannot be found from symmetries of the $d = 9$ theory. Instead, we find the type-II T-duality rules as follows. As we have seen in the previous sections, the compactification

⁹ It is not always the case that a continuous transformation of a T-duality group is a particular gauge transformation in a higher-dimensional theory. The simplest counter-example is provided by considering the coupling of the type-I string to one Abelian vector multiplet. The T-duality symmetry in nine dimensions is extended from $O(1,1)$ to $O(2,1)$. $SO(2,1)$ has several discrete transformations that take us from the sheet of $O(2,1)$ which is connected to the identity to other sheets. Each of them generates a \mathbb{Z}_2 subgroup. One of them is Buscher's T-duality. Each sheet of $O(2,1)$, and, in particular, the one connected to the identity, is three dimensional: one transformation is a special g.c.t. transformation in $d = 10$, another corresponds to a special $U(1)$ gauge transformation in $d = 10$ but the third one yields a non-trivial solution-generating transformation in $d = 10$ [2]. The effect of this transformation is to convert uncharged solutions into charged ones. For more details about this case, see Ref. [30].

Table 2

The weights w of the fields of $d = 9$ type-11 supergravity under the global $O(1, 1)$ symmetry

Field	Weight w	Field	Weight w
k	1	B	1
$A^{(1)}$	$-1/2$	$A^{(2)}$	-1
$\hat{B}^{(1)}$	0	$\hat{B}^{(2)}$	$1/2$
ℓ	$1/2$	c	$-1/2$

of both the ten-dimensional type-IIA and type-IIB theories lead to the same nine-dimensional supergravity theory. Therefore, the same nine-dimensional field configuration can be embedded in a ten-dimensional theory (or “decompactified”) in two different ways¹⁰ yielding two different ten-dimensional field configurations of two different theories.

Using the two inequivalent embeddings given in Eqs. (30) and (67) one finds that the transformation rules for the type-II duality symmetry that maps the type-IIB superstring onto the type-IIA superstring is given by

$$\begin{aligned}
 \hat{C}_{\underline{x}\mu\nu} &= \frac{2}{3} \left[\hat{B}_{\mu\nu}^{(2)} + 2\hat{B}_{\underline{x}[\mu}^{(2)}\hat{\gamma}_{\nu]\underline{x}}/\hat{\gamma}_{\underline{x}\underline{x}} \right], \\
 \hat{C}_{\mu\nu\rho} &= \frac{8}{3}\hat{D}_{\underline{x}\mu\nu\rho} + \epsilon^{ij}\hat{B}_{\underline{x}[\mu}^{(i)}\hat{B}_{\nu\rho]}^{(j)} + \epsilon^{ij}\hat{B}_{\underline{x}[\mu}^{(i)}\hat{B}_{\underline{x}]\nu}^{(j)}\hat{\gamma}_{\rho]\underline{x}}/\hat{\gamma}_{\underline{x}\underline{x}}, \\
 \hat{g}_{\mu\nu} &= \hat{\gamma}_{\mu\nu} - \left(\hat{\gamma}_{\underline{x}\mu}\hat{\gamma}_{\underline{x}\nu} - \hat{B}_{\underline{x}\mu}^{(1)}\hat{B}_{\underline{x}\nu}^{(1)} \right) / \hat{\gamma}_{\underline{x}\underline{x}}, \\
 \hat{B}_{\mu\nu}^{(1)} &= \hat{B}_{\mu\nu}^{(1)} + 2\hat{B}_{\underline{x}[\mu}^{(1)}\hat{\gamma}_{\nu]\underline{x}}/\hat{\gamma}_{\underline{x}\underline{x}},
 \end{aligned} \tag{85}$$

$$\begin{aligned}
 \hat{g}_{\underline{x}\mu} &= \hat{B}_{\underline{x}\mu}^{(1)}/\hat{\gamma}_{\underline{x}\underline{x}}, & \hat{B}_{\underline{x}\mu}^{(1)} &= \hat{\gamma}_{\underline{x}\mu}/\hat{\gamma}_{\underline{x}\underline{x}} \\
 \hat{A}_{\mu}^{(1)} &= -\hat{B}_{\underline{x}\mu}^{(2)} + \hat{\ell}\hat{B}_{\underline{x}\mu}^{(1)}, & \hat{g}_{\underline{x}\underline{x}} &= 1/\hat{\gamma}_{\underline{x}\underline{x}}, \\
 \hat{\phi} &= \hat{\varphi} - \frac{1}{2}\log(-\hat{\gamma}_{\underline{x}\underline{x}}), & \hat{A}_{\underline{x}}^{(1)} &= \hat{\ell}.
 \end{aligned} \tag{86}$$

Similarly, the type-11 duality map from the type-IIA onto the type-IIB superstring is given by

$$\begin{aligned}
 \hat{D}_{\underline{x}\mu\nu\rho} &= \frac{3}{8} \left[\hat{C}_{\mu\nu\rho} - \hat{A}_{[\mu}^{(1)}\hat{B}_{\nu\rho]}^{(1)} + \hat{g}_{\underline{x}[\mu}\hat{B}_{\nu\rho]}^{(1)}\hat{A}_{\underline{x}}^{(1)}/\hat{g}_{\underline{x}\underline{x}} - \frac{3}{2}\hat{g}_{\underline{x}[\mu}\hat{C}_{\nu\rho]\underline{x}}/\hat{g}_{\underline{x}\underline{x}} \right], \\
 \hat{\gamma}_{\mu\nu} &= \hat{g}_{\mu\nu} - \left(\hat{g}_{\underline{x}\mu}\hat{g}_{\underline{x}\nu} - \hat{B}_{\underline{x}\mu}^{(1)}\hat{B}_{\underline{x}\nu}^{(1)} \right) / \hat{g}_{\underline{x}\underline{x}}, \\
 \hat{B}_{\mu\nu}^{(2)} &= \frac{3}{2}\hat{C}_{\mu\nu\underline{x}} - 2\hat{A}_{[\mu}^{(1)}\hat{B}_{\nu]\underline{x}}^{(1)} + 2\hat{g}_{\underline{x}[\mu}\hat{B}_{\nu]\underline{x}}^{(1)}\hat{A}_{\underline{x}}^{(1)}/\hat{g}_{\underline{x}\underline{x}}, \\
 \hat{B}_{\mu\nu}^{(1)} &= \hat{B}_{\mu\nu}^{(1)} + 2\hat{g}_{\underline{x}[\mu}\hat{B}_{\nu]\underline{x}}^{(1)}/\hat{g}_{\underline{x}\underline{x}}, & \hat{\gamma}_{\underline{x}\mu} &= \hat{B}_{\underline{x}\mu}^{(1)}/\hat{g}_{\underline{x}\underline{x}}, \\
 \hat{B}_{\underline{x}\mu}^{(1)} &= \hat{g}_{\underline{x}\mu}/\hat{g}_{\underline{x}\underline{x}}, & \hat{B}_{\underline{x}\mu}^{(2)} &= -\hat{A}_{\mu}^{(1)} + \hat{A}_{\underline{x}}^{(1)}\hat{g}_{\underline{x}\mu}/\hat{g}_{\underline{x}\underline{x}}, \\
 \hat{\gamma}_{\underline{x}\underline{x}} &= 1/\hat{g}_{\underline{x}\underline{x}}, & \hat{\ell} &= \hat{A}_{\underline{x}}^{(1)}, \\
 \hat{\varphi} &= \hat{\phi} - \frac{1}{2}\log(-\hat{g}_{\underline{x}\underline{x}}).
 \end{aligned} \tag{87}$$

The dual of the type-IIA metric $\hat{g}_{\underline{x}\underline{x}}$ is given by the inverse of the type-IIB metric $\hat{\gamma}_{\underline{x}\underline{x}}$ and vice versa. For a torus compactification this means that the usual $R \rightarrow \alpha'/R$ duality is replaced by the map $R_{IIA} \rightarrow \alpha'/R_{IIB}$ where R_{IIA} is the torus radius characterizing

¹⁰ One way of embedding is given in (30) while the other way is given in (67)

the type-IIA decompactification and R_{IIB} is the torus radius characterizing the type-IIB decompactification, as in Refs. [6,7].

We observe that the type-II T-duality rules are a true generalization of Buscher's duality rules [10] in the sense that if we set the **type-IIA** and type-III3 Ramond-Ramond fields to zero and identify the remaining NS-NS **type-IIA** and type-IIB fields with the type-1 fields, the above rules reduce to (83). Furthermore, note that the type-II duality rule is a non-trivial solution-generating transformation in the following sense: given a solution of the type-IIA string equations of motion with one isometry, it generates a solution of the type-III3 equations of motion and vice versa.

This type-II T-duality maps the symmetries of each individual ten-dimensional type-II theory into the other. This is specially useful when one symmetry is manifest in one theory but not in the other. This is the case of $SL(2, \mathbb{R})$, which is manifest in the type-IIB theory (with or without isometries) but it is not manifest by any means in the type-IIA theory (with one **isometry**). The reader can check that the type-IIB S-duality rules Eqs. (79) are mapped into the type-IIA S-duality rules Eqs. (77) by the type-II T-duality rules Eqs. (86).

The analysis we have given for the bosonic sector can be straightforwardly extended to the full supersymmetric theory with fermions, since the non-compact symmetries of the bosonic sector are known to extend to symmetries of the full supergravity theory. Of particular interest are supersymmetric solutions which admit Killing spinors, and we now address the question of whether the image of a supersymmetric solution under duality is again supersymmetric. For example, the xy-duality transformations are simple coordinate transformations in eleven dimensions and, therefore, they preserve eleven-dimensional unbroken supersymmetries. If the eleven-dimensional Killing spinors corresponding to a given solution are **independent** of the coordinates x and y , they will be invariant under this duality transformation. Under these conditions, upon compactification of the coordinates x or y or a combination of both, we will get ten-dimensional unbroken supersymmetries. The Killing spinors will depend on which coordinate we have **compactified** and the different choices will be related by xy-duality transformations in ten dimensions. On the other hand, if the eleven-dimensional Killing spinors depend on x or y we expect that supersymmetry will be broken by ny-duality, as in the case studied in Ref. [26]. We have seen that the type-II T-duality rules do not correspond to any symmetry at all in nine dimensions. Therefore, all nine-dimensional properties will be preserved, in particular unbroken supersymmetries. Again everything depends on the preservation of the Killing spinors in the compactification procedure. Ten-dimensional Killing spinors with explicit dependence on the direction with respect to which we are going to dualize will lead to broken supersymmetry while duality will commute with the space-time supersymmetry if the Killing spinors are independent of the duality direction.

The type-IIA S-duality rules are based on the existence of two isometries corresponding to the directions x and y . It is interesting to note that transformations based on the existence of two isometries in the higher-dimensional theory have been considered before, albeit in a slightly different context, in the construction of the Kaluza-Klein or

Gross-Perry-Sorkin (GPS) magnetic **monopole** [31]¹¹. In essence, in the GPS case one considers a five-dimensional configuration with two isometries and “compactifies” alternatively the two corresponding directions getting two four-dimensional configurations each of them with a different isometry (the Euclidean Taub-NUT solution and the GPS magnetic monopole).

In our language we could say that these two configurations are *dual*. There are only a few inessential differences between the GPS case and our case:

(i) The original higher dimensional theory.

(ii) The fact that in the GPS case one of the isometry directions is time-like and the other one is space-like while in our case both isometry directions are space-like. The compactification of a time-like direction leads to a four-dimensional Euclidean **Kaluza-Klein** theory with a vector field and a scalar. In order to avoid the occurrence of the vector field one has to impose more restrictive conditions on the higher-dimensional configurations: they must be not just time-independent (stationary) but *static*¹². The presence of the unwanted scalar can be avoided by choosing five-dimensional configurations as those considered in Refs. [31] with $g_{00}^{(5)} = 1$.

6. Duality in eleven dimensions

The eleven-dimensional supergravity theory has no duality symmetries of its equations of motion for general backgrounds. For backgrounds with one isometry, there should be an $SO(1,1)$ symmetry of the equations of motion corresponding to the s-duality of the type-IIA theory; this is essentially a particular eleven-dimensional diffeomorphism. For backgrounds with two isometries, there should be an $SO(1,1) \times SL(2, \mathbb{R})$ symmetry of the equations of motion corresponding to the duality symmetries of the $d = 9$ theory. We have already identified an $O(2)$ subgroup of $SL(2, \mathbb{R})$ as rotations and reflections in the xy plane. It is clear that there cannot be an analogue of the \mathbb{Z}_2 T-duality symmetry here as the type-IIB supergravity theory cannot be obtained from any eleven-dimensional theory. However, if we restrict ourselves to the *subset* of solutions of $N = 1$, $d = 11$ supergravity which have two commuting isometries in the directions parametrized by the coordinates $y = x^{10}$ and $x = x^9$ and which, in addition, satisfy

$$\hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}} = \hat{g}_{\hat{\mu}\hat{\nu}} - 0. \quad (88)$$

then the configuration of $N = 1$, $d = 11$ gives a solution of $N = 1$, $d = 10$ supergravity with one isometry upon dimensional reduction, and this has a \mathbb{Z}_2 **Buscher** duality symmetry. The algebraic constraints are then the truncation from type-IIA supergravity to $N = 1$, $d = 10$ supergravity equation (25) written in eleven dimensions and the T-duality rules can be rewritten in eleven-dimensional form:

¹¹ We note that recently a six-brane solution of eleven-dimensional **supergravity** has been constructed which is an exact analogue in eleven dimensions of the GPS **monopole** in five dimensions [12].

¹² The time-like Killing vector is then “hypersurface-orthogonal” which in practice means that all the elements $g_{0i}^{(5)}$ of the five-dimensional metric can be made to vanish in an appropriate coordinate system.

$$\begin{aligned}
\hat{C}'_{\mu\nu\bar{y}} &= \hat{C}_{\mu\nu\bar{y}} + 2\hat{g}_{\bar{x}[\mu}\hat{C}_{\nu]\bar{x}\bar{y}}/\hat{g}_{\bar{x}\bar{x}}, & \hat{C}'_{\mu\bar{x}\bar{y}} &= -\frac{2}{3}\hat{g}_{\bar{x}\mu}/\hat{g}_{\bar{x}\bar{x}}, \\
\hat{g}'_{\mu\nu} &= \left(-\hat{g}_{\bar{y}\bar{y}}\right)^{\frac{1}{6}}\left(-\hat{g}_{\bar{x}\bar{x}}\right)^{\frac{1}{3}}\left\{\hat{g}_{\mu\nu} - \hat{g}_{\bar{x}\mu}\hat{g}_{\bar{x}\nu}/\hat{g}_{\bar{x}\bar{x}} - \frac{9}{4}\hat{C}_{\mu\bar{x}\bar{y}}\hat{C}_{\nu\bar{x}\bar{y}}/\left(\hat{g}_{\bar{x}\bar{x}}\hat{g}_{\bar{y}\bar{y}}\right)\right\}, \\
\hat{g}'_{\bar{x}\mu} &= \frac{3}{2}\left(-\hat{g}_{\bar{y}\bar{y}}\right)^{-\frac{1}{6}}\left(-\hat{g}_{\bar{x}\bar{x}}\right)^{-\frac{2}{3}}\hat{C}_{\mu\bar{x}\bar{y}}, & \hat{g}'_{\bar{y}\bar{y}} &= -\left(-\hat{g}_{\bar{y}\bar{y}}\right)^{\frac{2}{3}}\left(-\hat{g}_{\bar{x}\bar{x}}\right)^{-\frac{2}{3}}, \\
\hat{g}'_{\bar{x}\bar{x}} &= -\left(-\hat{g}_{\bar{y}\bar{y}}\right)^{-\frac{5}{6}}\left(-\hat{g}_{\bar{x}\bar{x}}\right)^{-\frac{2}{3}}.
\end{aligned} \tag{89}$$

The condition $\hat{g}_{\bar{\mu}\bar{y}} = 0$ means that the Killing vector $\partial/\partial\bar{y}$ is hypersurface-orthogonal, i.e. orthogonal to the hypersurfaces of constant value of \bar{y} . The eleven-dimensional manifold \mathcal{M}^{11} is the product of a ten-dimensional manifold times a circle $\mathcal{M}^{11} = \mathcal{M}^{10} \times S^1$.

It is interesting to see what the membrane analogue is of the usual $R \rightarrow 1/R$ duality¹³. For this purpose we consider a membrane moving in the space $\mathcal{M}^9 \times T^2$ (nine-dimensional Minkowski space-times a two-torus) and assume that the radius of the two-torus in the $x^2 = x$ direction is R_1 , i.e. $\hat{g}_{\bar{x}\bar{x}} = -(R_1)^2$ and similarly that the radius in the $x^{10} = y$ direction is R_2 , i.e. $\hat{g}_{\bar{y}\bar{y}} = -(R_2)^2$. We find that for this case the duality rules are given by

$$\begin{aligned}
R'_1 &= 1/\left(R_1^{2/3}R_2^{5/6}\right), & R'_2 &= (R_2/R_1)^{2/3}, \\
\hat{\eta}'_{\mu\nu} &= R_1^{2/3}R_2^{1/3}\hat{\eta}_{\mu\nu}.
\end{aligned} \tag{90}$$

It is well known that in case of the string duality the one-torus with the self-dual radius $R = 1$ is special in the sense that symmetry-enhancement occurs. We find that in the case of the membrane there is a whole one-parameter family of two-tori which are self-dual. They are characterized by the following radii:

$$R_1 = R, \quad R_2 = \frac{1}{R^2}. \tag{91}$$

It would be interesting to see in which sense this family of two tori plays a special role in membrane dynamics.

7. Examples

As an illustration of our results we shall now apply the duality transformations constructed in previous sections to generate new solutions of the type-IIA, type-IIB and eleven-dimensional supergravity theories. Our starting point will be the “Supersymmetric String Waves” (SSW) of Ref. [15] which are solutions of the heterotic string and also of the type-II equations of motion. Under type-I T-duality they are dual to the “Generalized Fundamental Strings” (GFS) solutions of Refs. [16,17].

¹³ For simplicity, we assume from now on that all radii and fields have been redefined to be dimensionless, as in Ref. [1].

Both the SSW and GFS solutions can be embedded into the type-I, type-IIA and type-IIB theories. We will denote the embedded solutions by SSW, SSW(A) and SSW(B) respectively and similarly for the GFS. We start with SSW(A) and GFS(A) and we first perform a discrete xy-duality transformation using Eqs. (77). The xy-duality generates new solutions of the type-IIA equations of motion which we denote by SSW(A') and GFS(A'), respectively. Next, we perform a type-II T-duality transformation to the type-IIB theory according to Eqs. (87). This leads to new solutions of the type-IIB theory where SSW solutions are converted into GFS solutions and vice versa. We denote these new solutions by GFS(B') and SSW(B'), respectively. Finally, we perform a further xy-duality transformation using Eqs. (79) getting GFS(B) and SSW(B). The reader may check that the GFS(B) and SSW(B) solutions are related by the type-II T-duality Eqs. (86) to the original SSW(A) and GFS(A) solutions we started from, as they should. Below we give the explicit form of the new solutions obtained in this manner.

7.1. Duality rotation of SSW

We first consider the SSW case. The fields of the SSW(A) solution are given by

$$\text{SSW(A)} \left\{ \begin{array}{l} ds^2 = 2 (du + \mathcal{A}_u du + 2\mathcal{A}_i dx^i) du - dx^i dx^i, \\ \hat{B}^{(1)} = 2\mathcal{A}_i dx^i \wedge du, \\ \hat{\phi} = 0. \end{array} \right. \quad (92)$$

The indices i, j run from 1 to 8 and $u = \frac{1}{\sqrt{2}}(t + x)$, $v = \frac{1}{\sqrt{2}}(t - x)$. Here \mathcal{A}_u and \mathcal{A}_i are arbitrary functions, independent of u and v , that satisfy the equations

$$\Delta \mathcal{A}_u = 0, \quad \Delta \partial^{[i} \mathcal{A}^{j]} = 0, \quad (93)$$

where the Laplacian is taken over the eight transverse directions only.

Performing the xy-duality transformations Eqs. (77) we get the new SSW(A') solution:

$$\text{SSW(A')} \left\{ \begin{array}{l} ds^2 = e^{-\frac{2}{3}\hat{\phi}} \left(dt + \frac{1}{\sqrt{2}} \mathcal{A}_i dx^i \right)^2 - e^{-\frac{2}{3}\hat{\phi}} (dx^2 + dx^i dx^i) \\ \hat{B}^{(1)} = -\frac{1}{\sqrt{2}} \mathcal{A}_i dx^i \wedge dx, \\ \hat{C} = \frac{\sqrt{2}}{3} \mathcal{A}_i dx^i \wedge dt \wedge dx, \\ \hat{A}^{(1)} = -e^{-\frac{4}{3}\hat{\phi}} \left\{ \left(1 - e^{\frac{4}{3}\hat{\phi}} \right) dt + \frac{1}{\sqrt{2}} \mathcal{A}_i dx^i \right\} \\ \hat{\phi} = \frac{3}{4} \log(1 - \mathcal{A}_u), \end{array} \right. \quad (94)$$

Next, we perform the type-II T-duality transformation Eqs. (87) and get the new GFS(B') solution

$$\text{GFS(B')} \left\{ \begin{array}{l} ds^2 = 2e^{-\hat{\phi}} (du + \mathcal{A}_i dx^i) dv - e^{\hat{\phi}} dx^i dx^i, \\ \hat{B}^{(2)} = e^{-2\hat{\phi}} (1 - e^{\hat{\phi}}) \mathcal{A}_i dx^i \wedge dv, \\ \hat{\phi} = \frac{1}{2} \log(1 - \mathcal{A}_u), \end{array} \right. \quad (95)$$

with all other fields vanishing.

Finally, an xy-duality transformation (Eqs. (79)) yields the following GFS (B) solution:

$$\text{GFS(B)} \begin{cases} ds^2 = 2e^{2\hat{\phi}} (\mathbf{dv} + \mathcal{A}_i dx^i) \mathbf{du} - dx^i dx^i, \\ \hat{\mathcal{B}}^{(1)} = e^{2\hat{\phi}} \left[\left(1 - e^{-\frac{1}{2}\hat{\phi}}\right) dv + \mathcal{A}_i dx^i \right] \mathbf{Adu}, \\ \hat{\phi} = -\frac{1}{2} \log(1 - \mathcal{A}_u). \end{cases} \quad (96)$$

This solution is just the original GFS solution but embedded into the type-IIB theory. Therefore, a further type-II T-duality transformation will take us back to the original SSW embedded into the type-IIA theory, i.e. the SSW(A) solution we started from.

7.2. Duality rotation of the GFS

We next consider the different duality rotations of the GFS solution. We start from the embedding into the type-IIA theory, i.e. the GFS(A) solution. It is given by

$$\text{GFS(A)} \begin{cases} ds^2 = 2e^{2\hat{\phi}} (\mathbf{dv} + \mathcal{A}_i dx^i) \mathbf{du} - dx^i dx^i, \\ \hat{\mathcal{B}}^{(1)} = 2e^{2\hat{\phi}} \left[\left(1 - e^{-2\hat{\phi}}\right) dv + \mathcal{A}_i dx^i \right] \wedge du, \\ \hat{\phi} = -\frac{1}{2} \log(1 - \mathcal{A}_u). \end{cases} \quad (97)$$

Performing the xy-duality transformations (77) we get the new solution

$$\text{GFS(A')} \begin{cases} ds^2 = e^{2\hat{\phi}} \left\{ \left(dt + \frac{1}{\sqrt{2}} \mathcal{A}_i dx^i \right)^2 - dx^2 \right\} - dx^i dx^i, \\ \hat{\mathcal{B}}^{(1)} = -e^{2\hat{\phi}} \left[\left(1 - e^{-2\hat{\phi}}\right) dt \wedge dx + \frac{1}{\sqrt{2}} \mathcal{A}_i dx^i \wedge dx \right], \\ \hat{\mathcal{C}} = \frac{\sqrt{2}}{3} e^{-2\hat{\phi}} \mathcal{A}_i dx^i \wedge \mathbf{dt} \mathbf{A} dx, \\ \hat{\phi} = -\frac{1}{2} \log(1 - \mathcal{A}_u). \end{cases} \quad (98)$$

We next apply the type-II T-duality rotation (87) and get the following SSW(B') solution:

$$\text{SSW(B')} \begin{cases} ds^2 = 2 (\mathbf{du} + \mathcal{A}_u dv + 2\mathcal{A}_i dx^i) \mathbf{dv} - dx^i dx^i, \\ \hat{\mathcal{B}}^{(2)} = 2\mathcal{A}_i dx^i \mathbf{A} dv, \\ \hat{\phi} = 0. \end{cases} \quad (99)$$

Finally, a further xy-duality transformation (79) gives the solution

$$\text{SSW(B)} \begin{cases} ds^2 = 2 (\mathbf{dv} + \mathcal{A}_u du + 2\mathcal{A}_i dx^i) \mathbf{du} - dx^i dx^i, \\ \hat{\mathcal{B}}^{(1)} = 2\mathcal{A}_i dx^i \mathbf{A} du, \\ \hat{\phi} = 0 \end{cases} \quad (100)$$

which is exactly what one should have expected: the original SSW solutions embedded into the type-IIB theory.

Note that the above examples do not exhaust the possible new solutions that can be built out of the GFS and the SSW. It would be of interest to apply the type-II S- and T-dualities to the various p-brane solutions of ten-dimensional supergravity and to investigate which solutions are related to each other by some combination of dualities and which solutions are independent ones.

7.3. Eleven-dimensional solutions

We finally consider the lifting of the SSW and GFS solutions to solutions of the eleven-dimensional theory. These liftings lead to solutions of eleven-dimensional supergravity which correspond to a supersymmetric string wave solution and a generalized fundamental membrane¹⁴ solution which we denote by SSW_{11} and GFM_{11} , respectively. The explicit form of the SSW_{11} and GFM_{11} solutions is given by

$$\text{SSW}_{11} \left\{ \begin{array}{l} ds^2 = 2 (\mathbf{du} + \mathcal{A}_u du + 2\mathcal{A}_i dx^i) \mathbf{du} - dx^i dx^i - dy dy \\ \hat{C} = \frac{4}{3} \mathcal{A}_i dx^i \wedge du \wedge dy \end{array} \right. , \quad (101)$$

and

$$\text{GFM}_{11} \left\{ \begin{array}{l} ds^2 = (1 - \mathbf{A}_i)^{-\frac{2}{3}} [2 (dv + \mathcal{A}_i dx^i) \mathbf{du} - dy dy] \\ \quad - (1 - \mathcal{A}_u)^{\frac{1}{3}} dx^i dx^i, \\ \hat{C} = \frac{4}{3} (1 - \mathbf{A}_i)^{-\frac{1}{3}} (\mathcal{A}_u dv + \mathcal{A}_i dx^i) \wedge du \wedge dy \end{array} \right. , \quad (102)$$

We note that the SSW_{11} solution is a generalization of the pp-wave solution of [18] containing the additional eight functions \mathcal{A}_i while the GFM_{11} solution generalizes the fundamental membrane solution of [19]. One may verify that the SSW_{11} given in Eqs. (101) is related to the GFM_{11} solution given in Eqs. (102) by the eleven-dimensional type-1 T-duality rules (89). Finally, it would be of interest to apply the $d=11$ type-1 T-duality to other eleven-dimensional solutions such as the p-brane solutions of [33].

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¹⁴ The reinterpretation of the ten-dimensional string solution as an eleven-dimensional (extreme) membrane solution was discussed in [32]. The duality transformations given in this paper only concern the source-free field equations. We will not discuss here the possible source terms and their duality transformations.

Appendix A. Conventions

We use double hats for eleven-dimensional objects, single hats for ten-dimensional objects and no hats for nine-dimensional objects. Greek or underlined indices are world indices, and Latin or non-underlined indices are Lorentz indices. We use the indices $\hat{\mu} = (\hat{\mu}, y) = (\mu, x, y)$, with $y = x^{10}$ and $x = x^9$. Our signature is $(+ - - \dots -)$. The antisymmetric Levi-Civita tensor $\hat{\epsilon}$ is defined by

$$\hat{\epsilon}^{\hat{\mu}_0 \dots \hat{\mu}_{10}} = 1. \quad (\text{A.1})$$

Our spin connection ω (in d dimensions) is defined by

$$\omega_\mu{}^{ab}(\epsilon) = -e^{\nu[a}(\partial_\mu e_\nu{}^{b]} - \partial_\nu e_\mu{}^{b]) - e^{\rho[a} e^{\sigma b]}(\partial_\sigma e_{c\rho}) e_\mu{}^c. \quad (\text{A.2})$$

The curvature tensor corresponding to this spin connection field is defined by

$$R_{\mu\nu}{}^{ab}(\omega) = 2\partial_{[\mu}\omega_{\nu]}{}^{ab} - 2\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^b, \quad R(w) \equiv e^\mu{}_a e^\nu{}_b R_{\mu\nu}{}^{ab}(\omega). \quad (\text{A.3})$$

Although we do not use differential forms, sometimes we use the following convention: when indices are not shown explicitly, we assume that all of them are world indices and all of them are completely antisymmetrized in the obvious order. For instance

$$\hat{G} = \partial \hat{C} - 2\hat{H}^{(1)} \hat{A}^{(1)} \quad (\text{A.4})$$

means

$$\hat{G}_{\hat{\mu}\hat{\nu}\hat{\sigma}\hat{\rho}} = \partial_{[\hat{\mu}} \hat{C}_{\hat{\nu}\hat{\sigma}\hat{\rho}]} - 2\hat{H}_{[\hat{\mu}\hat{\nu}\hat{\rho}}^{(1)} \hat{A}_{\hat{\sigma}]}^{(1)}. \quad (\text{A.5})$$

Appendix B. Eleven- and nine-dimensional fields

Here we present the expression of the eleven-dimensional fields in terms of the nine-dimensional ones. The components of the eleven-dimensional metric are

$$\begin{aligned} \hat{g}_{\underline{y}\underline{y}} &= -k^{\frac{2}{3}} e^{\frac{4}{3}\phi}, & \hat{g}_{\underline{x}\underline{x}} &= -k^{\frac{2}{3}} e^{\frac{4}{3}\phi} (\ell^2 + k e^{-2\phi}), \\ \hat{g}_{\underline{y}\underline{x}} &= -\ell k^{\frac{2}{3}} e^{\frac{4}{3}\phi}, & \hat{g}_{\underline{\mu}\underline{\nu}} &= -k^{\frac{2}{3}} e^{\frac{4}{3}\phi} A_\mu^{(1)} A_\nu^{(1)} - \ell k^{\frac{2}{3}} e^{\frac{4}{3}\phi} A_\mu^{(2)} A_\nu^{(2)}, \\ \hat{g}_{\underline{\mu}\underline{x}} &= -\ell k^{\frac{2}{3}} e^{\frac{4}{3}\phi} A_\mu^{(1)} - k^{\frac{2}{3}} e^{\frac{4}{3}\phi} (\ell^2 + k e^{-2\phi}) A_\mu^{(2)}, \\ \hat{g}_{\underline{\mu}\underline{\nu}} &= -k^{-\frac{1}{3}} e^{-\frac{2}{3}\phi} g_{\mu\nu} - k^{\frac{2}{3}} e^{\frac{4}{3}\phi} A_\mu^{(1)} A_\nu^{(1)} - k^{\frac{2}{3}} e^{\frac{4}{3}\phi} (\ell^2 + k e^{-2\phi}) A_\mu^{(2)} A_\nu^{(2)} \\ &\quad - 2\ell k^{\frac{2}{3}} e^{\frac{4}{3}\phi} A_\mu^{(1)} A_\nu^{(2)}, \end{aligned} \quad (\text{B.1})$$

and the components of the eleven-dimensional three-form \hat{C} are

$$\begin{aligned} \hat{C}_{\underline{\mu}\underline{\nu}\underline{\rho}} &= C_{\mu\nu\rho}, & \hat{C}_{\underline{\mu}\underline{x}\underline{y}} &= -\frac{2}{3} B_\mu, \\ \hat{C}_{\underline{\mu}\underline{\nu}\underline{x}} &= \frac{2}{3} (B_{\underline{\mu}\underline{\nu}}^{(1)} + A_{[\underline{\mu}}^{(2)} B_{\underline{\nu}]}), & \hat{C}_{\underline{\mu}\underline{\nu}\underline{y}} &= \frac{2}{3} (B_{\underline{\mu}\underline{\nu}}^{(2)} - A_{[\underline{\mu}}^{(1)} B_{\underline{\nu}]}). \end{aligned} \quad (\text{B.2})$$

The inverse relations are

$$\begin{aligned}
 k &= \left(-\hat{g}_{yy} \right)^{-\frac{1}{4}} \Delta^{\frac{1}{2}}, & A_{\mu}^{(1)} &= \left(\hat{g}_{x\mu} \hat{g}_{yy} - \hat{g}_{xy} \hat{g}_{\mu y} \right) / \Delta, \\
 \ell &= \hat{g}_{xy} / \hat{g}_{yy}, & A_{\mu}^{(2)} &= \left(\hat{g}_{y\mu} \hat{g}_{xx} - \hat{g}_{xy} \hat{g}_{\mu x} \right) / \Delta, \\
 \phi &= \frac{1}{8} \log \left[\left(-\hat{g}_{yy} \right)^7 / \Delta^2 \right], & B_{\mu} &= \frac{3}{2} \hat{C}_{x\mu y}, \\
 B_{\mu\nu}^{(1)} &= \frac{3}{2} \left[\hat{C}_{\mu\nu y} \Delta + (\mu \leftrightarrow x) \right] / A, & B_{\mu\nu}^{(2)} &= \frac{3}{2} \left[\hat{C}_{\mu\nu x} \Delta + (\mu \leftrightarrow y) \right] / A.
 \end{aligned} \tag{B.3}$$

where

$$A = \underline{\underline{\mathcal{L}}} - \hat{g}_{xy}^2. \tag{B.4}$$

The expression of $g_{\mu\nu}$ in terms of the eleven-dimensional fields is not very enlightening and, in any case, it can be readily obtained from the above formulae.

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